

RATIONAL PONTRYAGIN CLASSES AND KILLING FORMS

ALLEN BACK

It is well known that equivariant isomorphism classes of K -vector bundles over the homogeneous space K/H are in one-to-one correspondence with equivalence classes of representations of H on the fiber V over the basepoint. Also, if the isotropy representation $\alpha: H \rightarrow \text{Aut}(V)$ extends to a homomorphism $\bar{\alpha}: K \rightarrow \text{Aut}(V)$, then it is easy to see that the associated equivariant bundle is trivial. The main purpose of this note is to use these observations together with the Chern-Weil theory of characteristic classes to prove the following.

Theorem 1. *If the Killing form of a compact Lie group K restricts to a multiple of the Killing form of H , then the first Pontryagin class of the tangent bundle of K/H is torsion.*

If H is a simple Lie group, then any two H -invariant forms are proportional. Consequently, the simplest case in which to apply Theorem 1 is:

Corollary 2. *If K is compact and H is simple, then the first rational Pontryagin class of K/H is trivial.*

We shall in general follow the conventions and notations of [1]. The notations \underline{K} for the Lie algebra of K , Ad_K for the adjoint action of K on \underline{K} , and ad_K for its derivative will also be used.

Since K is compact, choice of an Ad_K invariant metric on \underline{K} allows us to write $\underline{K} = \underline{H} + \underline{M}$ where \underline{M} is the orthogonal complement of \underline{H} . The representation Ad_K restricted to H leaves \underline{M} invariant, and gives rise to the tangent bundle β_1 of K/H . If we let β_2 and β_3 be the bundles associated to Ad_H and to Ad_K restricted to H respectively, then we clearly have $\beta_3 = \beta_1 + \beta_2$ topologically.

Obviously β_3 comes from a representation extending to K , so β_3 is trivial. Consequently the first Pontryagin class of β_2 is the negative of that of β_1 .

Proof of Theorem 1. Equip β_2 and β_3 with the K -invariant canonical connections of the first kind (i.e., the vector \bar{X} induced by the element X of \underline{M}

is horizontal at the base point). The Lie algebra valued curvature forms of β_2 and β_3 are then given by $\Omega_2(\bar{X}, \bar{Y}) = -\text{ad}_H([X, Y]_H)$ and $\Omega_3(\bar{X}, \bar{Y}) = -\text{ad}_K([X, Y]_H)$ respectively where $[X, Y]_H$ is the H component of $[X, Y]$ in $\underline{K} = \underline{H} + \underline{M}$.

Now the first Pontryagin form p_1 of a connection with $\text{End}(V)$ valued curvature form Ω is given by

$$p_1(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = c \text{Tr}_V[\Omega(X_1, X_2)\Omega(X_3, X_4) - \Omega(X_1, X_3)\Omega(X_2, X_4) + \Omega(X_1, X_4)\Omega(X_2, X_3)],$$

where c is a universal constant, and $\text{Tr}_V[f]$ is the trace of the element $f \in \text{End}(V)$.

For the bundle β_3 , $\text{Tr}_K[\Omega_3(\bar{X}, \bar{Y})\Omega_3(\bar{Z}, \bar{W})]$ is simply $\langle [X, Y]_H, [Z, W]_H \rangle_K$ where \langle, \rangle_K is the Killing form of K . So for β_3 ,

$$p_1(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) = c[\langle [X_1, X_2]_H, [X_3, X_4]_H \rangle_K - \langle [X_1, X_3]_H, [X_2, X_4]_H \rangle_K + \langle [X_1, X_4]_H, [X_2, X_3]_H \rangle_K].$$

Since β_3 is trivial, this form is exact.

But the first Pontryagin form of β_2 is given by the same expression except that all K -Killing forms \langle, \rangle_K are replaced by H -Killing forms \langle, \rangle_H . So if the Killing form of K restricts to a nonzero multiple of that of H , it is immediate that $p_1(\beta_2)$ is rationally trivial. If the Killing form of K restricts to zero on H , then the semisimple part of H is trivial, and all Pontryagin classes of K/H will be zero.

It is interesting to note that the proof of Theorem 1 may also be carried out using the natural Riemannian connections on the β_i .

Other examples to which Theorem 1 applies may be readily constructed; e.g., $SU(P)/(SU(m))^n$. More generally, if H_1 and H_2 are semisimple, then any $\text{Ad}_{H_1 \times H_2}$ invariant form on $\underline{H}_1 + \underline{H}_2$ will have \underline{H}_1 and \underline{H}_2 orthogonal to each other. Consequently we have

Corollary 3. *If H_1 and H_2 are simple and conjugate inside K , then $p_1(K/(H_1 \times H_2))$ will be torsion.*

One might also notice that the proof of Theorem 1 immediately generalizes to higher characteristic forms. Given $A_i \in \underline{K}$ ($1 \leq i \leq n$), define the "higher Killing form" $B_n(A_1, A_2, \dots, A_n)$ to be $\text{Tr}_K(\text{ad}_K A_1 \text{ad}_K A_2 \dots \text{ad}_K A_n)$ (or the symmetrized version). Then the argument of Theorem 1 will express the symmetric sum characteristic forms s_n (i.e. the image under the Chern-Weil homomorphism of the symmetric polynomial $X \rightarrow \text{Tr}(X^n)$) of β_2 and β_3 in terms of the higher Killing forms of H and K respectively. Hence

Theorem 4. *If the higher Killing form B_n of K restricts to a nonzero multiple of that of H , then the s_n -characteristic class of the stable normal bundle of K/H is torsion.*

References

- [1] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vols. I, II, Wiley, New York, 1963, 1969.
- [2] A. Borel & F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, *Amer. J. Math.* **80** (1958) 458–538.

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